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COMMENT

Fractal dimension from the back-scattering cross section

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Abstract. The feasibility of recovering the fractal dimension of self-similar fractals from the back-scattering measurements is demonstrated. For this purpose, the well known Sierpinski gasket is modelled as an ensemble of lossless dielectric beads, and the scattered field computations are carried out using the coupled dipole approximation procedure.

Recently, attention has been focused upon the ways and means of determining the *fractal dimension* of structures endowed with the appropriate attributes, which methods are sufficiently different from the use of counting statistics to establish either the radius of gyration [1] or the density-density correlation function [2]. One of these newer procedures consists of optically taking the spatial Fourier transform of the fractal structure: the theoretical feasibility of this technique has been recently reported by us elsewhere [3], and it has been experimentally implemented by Allain and Cloitre [4]. Because it has also been shown that the neutron diffraction technique can yield substantial information about the structure of fractals [5], there is no doubt that the electromagnetic field theory will increasingly begin to be used for probing them.

In this connection, it is well known that the scattered field carries the signature of the target being illuminated [6, 7]. In particular, it is widely held that the back-scattering cross section could be an effective discriminant for the inverse target problems for non-fractal scatterers [8, 9]. Should not the same considerations apply to targets possessed with fractal geometries, so that their fractal dimension may be experimentally determined from back-scattering measurements?

In order to address the feasibility of this contention, we concentrate here on the back-scattering response of the Sierpinski gasket modelled as a collection of nodes resting upon a regular triangular grid [3, 10]. Each node of the gasket is occupied by a small, lossless dielectric particle, all particles being identical. The particles are assumed to be small enough in size in relation to the wavelength of the incident radiation in the ambient medium (free space) that they can be replaced by point electric polarisabilities [11] for the scattered field computations. Furthermore, we have taken the particles to be spherical beads in order to avoid complications due to anisotropy. Since the beads interact with each other, multiple scattering [12] has to be considered; the consequent procedure is currently being labelled as the coupled dipole approximation [13].

The finite Sierpinski gasket of order L, $L \geqslant 1$, is composed of 3^L nodes arranged on a regular, planar, triangular grid, and is recursively specified by the nodal placement function $q_L(x, y)$ given by

$$q_L(x, y) = q_{L-1}(x, y) * g_L(x, y) \qquad L > 1 \tag{1}$$

where $*$ is the spatial convolution operation [14]; the generator $g_L(x, y)$ and the initiator $q_1(x, y)$ are given by

$$\begin{aligned} g_L(x, y) &= \delta\{x, y\} + \delta\{x - a2^{L-1}, y - b2^{L-1}\} + \delta(x - a2^{L-1}, y + b2^{L-1}\} \\ q_1(x, y) &= g_1(x, y) \end{aligned} \tag{2}$$

and $\delta\{\cdot\}$ is the Dirac delta function. The spherical beads occupy those sites $\{x, y\}$ where $q_L(x, y)$ is unity, and for computational ease the origin of the coordinate system for the gasket of order L is shifted to its centroid after it has been generated using (1) and (2). The aspect ratio b/a is such that $0 < \tan^{-1}(b/a) < \pi/2$, and each of the 3^L spherical beads in this arrangement has a radius d which is small enough so that no two of them every touch. The beads are assumed identical to each other and possess a dielectric constant ε. Since the structure of order $L+1$ thus formed contains three gaskets of order L which are half its size, its fractal (similarity) dimension is $\log(3)/\log(2) = 1.584\,96$.

If $\boldsymbol{E}_m$ illuminates the mth bead, $m = 1, 2, 3, \ldots, 3^L$, an electric dipole moment [11]

$$\boldsymbol{p}_m = \alpha \boldsymbol{E}_m = 4\pi\varepsilon_0 d^3 (\varepsilon - \varepsilon_0)(\varepsilon - 2\varepsilon_0)^{-1} \boldsymbol{E}_m \tag{3}$$

is induced at its location $\boldsymbol{r}_m$, with α being the isotropic electric polarisability of the spherical beads. This induced dipole moment then re-radiates, the re-radiated field being given as [15]

$$4\pi\varepsilon_0 \boldsymbol{E}_{\mathrm{rad},m}(\boldsymbol{r}) = [k^2(\boldsymbol{n}_m \times \boldsymbol{p}_m) \times \boldsymbol{n}_m + [3\boldsymbol{n}_m(\boldsymbol{n}_m \cdot \boldsymbol{p}_m) - \boldsymbol{p}_m](R_m^{-2} - jkR_m^{-1})] \exp(jkR_m)/R_m \tag{4}$$

where

$$R_m = |\boldsymbol{r} - \boldsymbol{r}_m| \qquad \boldsymbol{n}_m = (\boldsymbol{r} - \boldsymbol{r}_m)/R_m. \tag{5}$$

A magnetic dipole is also induced on each of the spherical beads, as also are the higher-order multipoles [15]; however, the bead size parameter kd is assumed to be sufficiently small that (3) and (4) suffice to describe the scattering response of the mth bead.

The electromagnetic field incident on the gasket, $\boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{r})$, can be any arbitrary field so long as its source is not located anywhere inside or on the minimum sphere circumscribing the gasket. But the field $\boldsymbol{E}_m$ actually incident on the mth bead is not $\boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{r}_m)$ alone; it also consists of the fields $\boldsymbol{E}_{\mathrm{rad},n}(\boldsymbol{r}_m)$, $\forall n \neq m$, re-radiated by all of the other beads as well [12, 13]. It is then easy to see that the field exciting the mth bead can be self-consistently written as

$$\boldsymbol{E}_m = \boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{r}_m) + \sum_{n, n \neq m} \boldsymbol{E}_{\mathrm{rad},n}(\boldsymbol{r}_m). \tag{6}$$

Using, now, (3) and (4) in (6), the system of 3^{L+1} simultaneous equations [13]

$$\boldsymbol{E}_m = \boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{r}_m) + \alpha k^2 (4\pi\varepsilon_0)^{-1} \sum_{n, n \neq m} \{R_{mn}^{-1} \exp(jkR_{mn})[g_{mn}\boldsymbol{n}_{mn}(\boldsymbol{n}_{mn} \cdot \boldsymbol{E}_n) - h_{mn}\boldsymbol{E}_n]\} \tag{7}$$

must be solved, in order to obtain the various exciting fields $\boldsymbol{E}_m$. In (7),

$$R_{mn}=|\boldsymbol{r}_m-\boldsymbol{r}_n| \qquad \boldsymbol{n}_{mn}=(\boldsymbol{r}_m-\boldsymbol{r}_n)/R_{mn}$$
$$g_{mn}=3(kR_{mn})^{-2}-3j(kR_{mn})^{-1}-1 \qquad h_{mn}=\tfrac{1}{3}(g_{mn}-2). \tag{8}$$

Once the solution of (7) has been obtained, the total scattered field outside the circumscribing sphere can be computed simply as [12]

$$4\pi\varepsilon_0\boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{r})=\alpha\sum_m\{[k^2(\boldsymbol{n}_m\times\boldsymbol{E}_m)\times\boldsymbol{n}_m$$
$$+[3\boldsymbol{n}_m(\boldsymbol{n}_m\cdot\boldsymbol{E}_m)-\boldsymbol{E}_m](R_m^{-2}-jkR_m^{-1})]\exp(jkR_m)/R_m\} \tag{9}$$

which, for $kr\to\infty$, can be simplified to

$$4\pi\varepsilon_0\boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{r})=\alpha k^2r^{-1}\exp(jkr)\sum_m\{\exp(-jk\boldsymbol{r}_m\cdot\boldsymbol{r}/r)[\boldsymbol{E}_m-\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{E}_m)/r^2]\}. \tag{10}$$

It is again emphasised here that, in deriving (9), there are no restrictions placed on the dimensions a and b of the triangular grid apart from that neither be zero; the only limitation here is that the radius d of each of the beads be sufficiently small so that its scattering response can be adequately described via (3) and (4). Thus while the treatment of the individual bead here is in the long-wavelength approximation [15], the overall size of the ensemble could even be in the high-frequency regime.

Equations (7) and (10) were programmed on a DEC VAX 11/730 minicomputer and the exciting fields $\boldsymbol{E}_m$ as well as the scattering pattern $\boldsymbol{F}(\theta,\varphi)=\lim_{kr\to\infty}[kr\exp(-jkr)]\boldsymbol{E}_{\mathrm{sc}}(r,\theta,\varphi)$ were computed for an incident plane wave travelling in the $+z$ direction, thus propagating normally to the $z=0$ plane with unit amplitude at the origin. This particular configuration was chosen because the wave is then incident normally on the whole gasket and not merely on any projection of it.

Shown in figure 1 are the plots of the normalised back-scattering cross section

$$\Im_L=(2\pi/ka)^2|\boldsymbol{F}(0,0)|^2 \tag{11}$$

as a function of the normalised frequency ka, for the gasket evolutionary levels L ranging from 1-4, when $a=b$ and $d=(0.1)(a^2+b^2)^{1/2}$. The incident plane wave is either $\boldsymbol{E}_{\mathrm{inc}}=\boldsymbol{i}\exp(jkz)$ or $\boldsymbol{E}_{\mathrm{inc}}=\boldsymbol{j}\exp(jkz)$; in either case, it is observed that $\Im_L$ is independent of the incident polarisation. This is because the spherical beads, modelled as electric dipoles, are very weak scatterers and the *depolarising* effect of the multiple scattering between the beads does not turn out to be markedly noticeable at such low frequencies: although ka ranges from 0.05-2.15 in this figure, kd, the size parameter of the beads, does not exceed 0.3.

From this figure it turns out that the normalised back-scattering cross section can be set down as

$$\Im_L(ka)\approx 3^{2[L-1]}c_1(ka)^4 \tag{12}$$

where c_1 is a constant to be computed from the back-scattering cross section when $L=1$; here, $c_1=1.6752\times10^{-5}$. Thus, the hierarchy of the evolutionary levels is preserved in $\Im_L$ as the exponent of 3 in the RHS of (12); the exponent 4 of ka in (12) merely denotes that the calculations were made by approximating the scatterers by point electric dipoles.

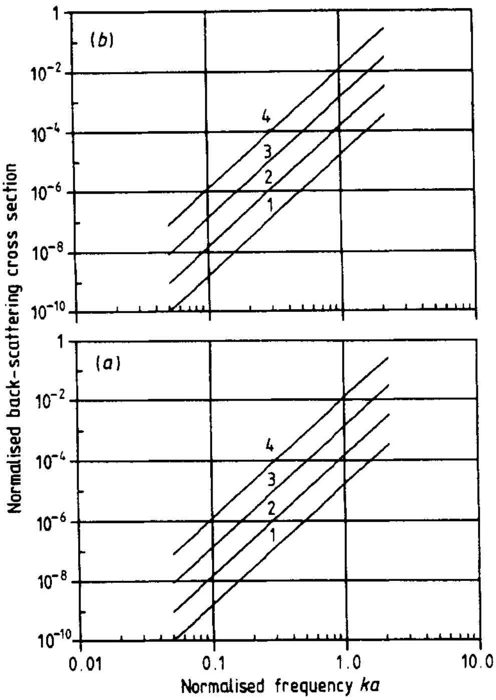

Figure 1. Computed values of $\Im_L(ka)$ for the Sierpinski gasket modelled as an ensemble of identical spherical beads. The values of L used are marked on the graphs. The gasket parameters $a = b$, while the radius of the beads, $d = \frac{1}{10}(a^2 + b^2)^{1/2}$. ($a$) $\boldsymbol{E}_{\text{inc}} = \mathrm{i}\exp(jkz)$, ($b$) $\boldsymbol{E}_{\text{inc}} = j\exp(jkz)$.

Let us now try to simulate a simple back-scattering measurement on the Sierpinski gasket. Suppose at a frequency ω_L only the gasket of order L is resolvable and $\Im_L$ measured. Next, the frequency is doubled to $\omega_{L+1} = 2\omega_L$ so that the gasket of the next higher order becomes resolvable. Taking into account that the overall dimension of the structure, and hence ka, remains constant it can be seen from (12) that $[\Im_{L+1}(ka)/\Im_L(ka)]^{1/2} = 3$ and the ratio $\frac{1}{2}\log[\Im_{L+1}(ka)/\Im_L(ka)]/\log(\omega_{L+1}/\omega_L)$ would be equal to the fractal dimension.

The extension of such an experiment to a rigorously self-similar fractal with an inherent scale factor s is obvious. Provided s is known *a priori*, measurements of $\Im$ should be made at two frequencies, one of which is a factor s times greater than the other, and the fractal dimension can then be deduced. If, on the other hand, s is not known, or if it is suspected that the fractal may not be rigorously self-similar, then measurements of $\Im$ should be made at discrete frequencies ω_i, $i = 1, 2, 3, \ldots$, spread over as wide a range as possible, and the successive ratios $\frac{1}{2}\log[\Im_{i+1}(ka)/\Im_i(ka)]/\log(\omega_{i+1}/\omega_i)$ should be plotted against i. Caution should be exercised in interpreting these ratios, however, because of the differences which exist between self-similar and self-affine fractals [16]. Should the fractal be strongly self-affine, i.e. it is possibly a perturbation of a self-similar fractal, then the computed ratios would not deviate significantly from their mean value, which can then be thought of as an experimentally measured fractal dimension. But if that is not the case and the fractal is merely self-affine, then the ratios should be thought of as a sequence of local fractal dimensions, possibly converging to some global fractal dimension, if any.

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